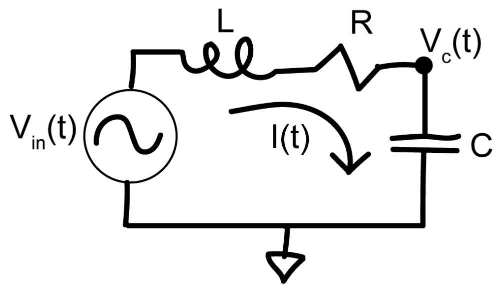
6 Week 6 - Spectral domain circuit analysis. Part 2

6.1 Circuits

Figure 1: RLC circuit with elements in series and driven by a sinusoid



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Let's work through an entire circuit problem. The application of Kirchoff's voltage law to a series RLC circuit yields

$$0 = -V_{in}(t) + L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}\int_{-\infty}^{t} dt'\; I(t'). \tag{6.6}$$

We put this in terms of the voltage drop across the capacitor, i.e., $V_C(t)$, so

$$I_C(t) = C\frac{dV_C(t)}{dt} \tag{6.7}$$

and

$$V_{in}(t) = LC\frac{d^2V_C(t)}{dt^2} + RC\frac{dV_C(t)}{dt} + V_C(t) \tag{6.8}$$

or

$$\omega_o^2 V_{in}(t) = \frac{d^2V_C(t)}{dt^2} + \frac{1}{\tau_L}\frac{dV_c(t)}{dt} + \omega_o^2 V_C(t) \tag{6.9}$$

recalling that

$$\tau_L \equiv L/R, \tag{6.10}$$

and

$$LC = 1/\omega_o^2. \tag{6.11}$$

Fourier transforming yields

$$\begin{aligned} \omega_o^2\tilde{V}_{in}(\omega) &= -\omega^2\tilde{V}_C(\omega) + \frac{i\omega}{\tau_L}\tilde{V}_C(\omega) + \omega_o^2\tilde{V}_C(\omega) \qquad (6.12)\\ &= \left[\omega_o^2 - \omega^2 + i\frac{\omega}{\tau_L}\right]\tilde{V}_C(\omega) \end{aligned}$$

or

$$
\begin{aligned}
\tilde{V}_C(\omega) &= \frac{1}{\omega_o^2 - \omega^2 + i\omega/\tau_L}\ \omega_o^2 \tilde{V}_{in}(\omega) \\
&= A\ \frac{1}{i}\sqrt{\frac{\pi}{2}}\frac{1}{\omega_o^2 - \omega^2 + i\omega/\tau_L}\ \omega_o^2\left[\delta(\omega-\omega_D)\ -\ \delta(\omega+\omega_D)\right] \\
&= A\ \frac{1}{i}\sqrt{\frac{\pi}{2}}\frac{\omega_o^2 - \omega^2 - i\omega/\tau_L}{(\omega_o^2 - \omega^2)^2 + (\omega/\tau_L)^2}\ \omega_o^2\left[\delta(\omega-\omega_D)\ -\ \delta(\omega+\omega_D)\right].
\end{aligned}
\tag{6.13}
$$

We now transform back to the time domain.

$$
\begin{aligned}
V(t) &= \frac{A\omega_o^2}{\sqrt{2\pi}}\frac{1}{i}\sqrt{\frac{\pi}{2}}\ \left[\int_{-\infty}^{\infty} d\omega\ \frac{\omega_o^2 - \omega^2 - i(\omega/\tau_L)}{(\omega_o^2 - \omega^2)^2 + (\omega/\tau_L)^2}\delta(\omega-\omega_D)e^{i\omega t}\right] \\
&- \frac{A\omega_o^2}{\sqrt{2\pi}}\frac{1}{i}\sqrt{\frac{\pi}{2}}\ \left[\int_{-\infty}^{\infty} d\omega\ \frac{\omega_o^2 - \omega^2 - i(\omega/\tau_L)}{(\omega_o^2 - \omega^2)^2 + (\omega/\tau_L)^2}\delta(\omega+\omega_D)e^{i\omega t}\right] \\
&= \frac{A\omega_o^2}{\sqrt{2\pi}}\frac{1}{i}\sqrt{\frac{\pi}{2}}\ \left[\frac{\omega_o^2 - \omega_D^2 - i(\omega_D/\tau_L)}{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}e^{i\omega_D t}\right] \\
&- \frac{A\omega_o^2}{\sqrt{2\pi}}\frac{1}{i}\sqrt{\frac{\pi}{2}}\ \left[\frac{\omega_o^2 - \omega_D^2 + i(\omega_D/\tau_L)}{(\omega_o^2 + \omega_D^2)^2 + (\omega_D/\tau_L)^2}e^{-i\omega_D t}\right] \\
&= \frac{A\omega_o^2}{\sqrt{2\pi}}\ 2\ \sqrt{\frac{\pi}{2}}\ \left[\frac{\omega_o^2 - \omega_D^2}{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}\frac{e^{i\omega_D t} - e^{-i\omega_D t}}{2i}\right] \\
&- \frac{A\omega_o^2}{\sqrt{2\pi}}\ 2\ \sqrt{\frac{\pi}{2}}\ \left[\frac{(\omega_D/\tau_L)}{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}\frac{e^{i\omega_D t} + e^{-i\omega_D t}}{2}\right] \\
&= A\omega_o^2\ \left[\frac{\omega_o^2 - \omega_D^2}{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}\sin(\omega_D t)\right] - A\omega_o^2\ \left[\frac{\omega_D/\tau_L}{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}\cos(\omega_D t)\right] \\
&= A\omega_o^2\ \frac{(\omega_o^2 - \omega_D^2)\sin(\omega_D t) - (\omega_D/\tau_L)\cos(\omega_D t)}{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2} \\
&= A\frac{\omega_o^2}{\sqrt{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}}\ \ \frac{(\omega_o^2 - \omega_D^2)\sin(\omega_D t) - (\omega_D/\tau_L)\cos(\omega_D t)}{\sqrt{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}}.
\end{aligned}
\tag{6.14}
$$

Note that $(\omega_o^2 - \omega_D^2)$, ω_D/τ_L, and $\sqrt{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}$ form a right triangle. Further, recall that

$$
\sin(\omega_D t\ -\ \phi) = \cos\phi\sin(\omega_D t) - \sin\phi\cos(\omega_D t). \tag{6.15}
$$

Then

$$
V_C(t)\ =\ A\frac{\omega_o^2}{\sqrt{(\omega_o^2 - \omega_D^2)^2 + (\omega_D/\tau_L)^2}}\sin\left[\omega_D t - \mathrm{atan}\left(\frac{\omega_\mathrm{D}/\tau_\mathrm{L}}{\omega_\mathrm{o}^2 - \omega_\mathrm{D}^2}\right)\right]. \tag{6.16}
$$